differentiate

$\frac{d}{dx}(5) = 0$

$\frac{d}{dx}(4x) = 4$

$\frac{d}{dx}(4x^2) = 8x$

$\frac{d}{dx}(\sin x) = \cos x$

$\int 0 \, dx = c$

$\int 4 \, dx = 4x + c$

$4x + 3$

$4x - 17$

$4x + 1/2$

Indefinite Integration - Classwork

1/5

Take a piece of notebook paper and cover up the paragraph under the chart below. Below, there are 5 terms. Write what you feel are the inverses of each of these terms.

term	5	1/3	boy	dog	hot dog
its inverse	−5	3, −1/3	girl	cat	cold cat

As ridiculous as the last problem is (how can you have an inverse of a hot dog?) so are they all ridiculous. The problem is that all of these terms aboves are nouns. Inverses refer not just to an opposite but to an opposite process. We take inverses of verbs, not nouns. Write the inverses of these processes.

process	sit down	get dressed	take a book home	get wet	go to sleep
its inverse	stand up				

f^{-1}

In mathematics, you have learned about operations on functions f. The inverse operation is denoted as f^{-1} (not to be confused with x^{-1}, the reciprocal of x (remember x is a noun and f is a process which is a verb.) When ever you perform an operation and immediately perform its inverse, you will end up exactly where you started with. We say that $f^{-1}[f(x)] = x$ and also $f[f^{-1}(x)] = x$. Below write some mathematical functions and their inverses.

Function	Inverse
$f = x^2$	$g = \sqrt{x}$
$f = x + 2$	$g = x - 2$

Function	Inverse
$\sin x$	$\arcsin x$

So, obviously since differentiation of functions is a process, we must have an inverse of that process. We call that process antidifferentiation. For instance, we know that the derivative of $y = x^3$ is $3x^2$. So it makes sense to say that an antiderivative of $3x^2$ is x^3.

However, it is important to say *an* antiderivative of $3x^2$ rather than *the* antiderivative of $3x^2$ for the simple reason that the derivative of $y = x^3$ is $3x^2$, but so is the derivative of $y = x^3 + 2$, $y = x^3 - 5$, and $y = x^3 + 6\pi$. There are an infinite number of functions whose derivative is $3x^2$. So when we go backwards to the antiderivative it is impossible to determine which function it came from. So to cover our bets, we say that the antiderivative of $3x^2$ is $x^3 + C$, where C represents a constant. We call C the constant of integration. It is important to attach the $+C$ after every antiderivative. What you are doing is saying that the antiderivative is a family of functions rather than one specific function.

$f(g(x)) = (\sqrt{x})^2 = x$

$f(g(x)) = x - 2 + 2 = x$

$g(f(x)) = \sqrt{x^2} = x$

$f = 2 + x$ $g = -2 + x$

$f = x + 2$

$y = x + 2$

$x = y + 2$

$x - 2 = y$

$y = \sin x$

The process of taking antiderivatives is called *integration*, specifically ***indefinite integration*** because of the constant of integration C. So we do not have to write the word antiderivative again, we use a symbol to represent an antiderivative. That symbol is called an integral sign which is written like this: $\int$. The way we write an integral is:

$\int \boxed{f(x)}\, dx = F(x) + C$. The dx tells you what the important variable is when you are integrating just as you need to know what the important variable is when you differentiate $\left(\frac{dy}{dt} \text{ as opposed to } \frac{dy}{dx}\right)$.

So, since $\frac{d}{dx}(4x) = 4$, we will say that $\int 4\, dx = 4x + C$ and

and since $\frac{d}{dx}(x^2 + 3x - 1) = 2x + 3$, we will say that $\int (2x+3)\, dx = x^2 + 3x + C$

Just as we have derivative rules, we have a corresponding rule for integrals. Here are some basic integration rules.

Differentiation formula	*Integration formula*
$\frac{d}{dx}[C]=0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx]=k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)]=kf'(x)$ a constant can be "factored out"	$\int kf(x)\,dx = k\int f(x)\,dx + C$ - factor out constant
$\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$ the derivative of a sum is the sum of derivatives	$\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx + C$ integral of a sum is the sum of the integrals
* $\frac{d}{dx}[x^n]=nx^{n-1}$ - the power rule	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ - the power rule reversed

Examples - find the integral of each of the following:

1) $\int 7dx = 7x + C$

2) $\int x^5 dx = \frac{1}{6}x^6 + C$

3) $\int x^{12} dx = \frac{1}{13}x^{13} + C$

$\int 7x^0\,dx$

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$\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$ the derivative of a sum is the sum of derivatives	$\int[f(x)\pm g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx + C$ integral of a sum is the sum of the integrals
$\frac{d}{dx}[x^n]=nx^{n-1}$ - the power rule	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ - the power rule reversed

4) $\int (x^4 - x^2)\,dx$

$= \frac{x^5}{5} - \frac{x^3}{3} + C$

5) $\int (t^3 + t + 1)\,dt$

$= \frac{1}{4}t^4 + \frac{1}{2}t^2 + t + C$

6) $\int 3x^3\,dx$

$= \frac{3}{4}x^4 + C$

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7) $\int (2x^2 - 7x - 8)\,dx$

$= \frac{2}{3}x^3 - \frac{7}{2}x^2 - 8x + C$

8) $\int \left(\frac{3}{4}x^5 + \frac{5}{3}x^2 - \frac{x}{2}\right) dx$

$= \frac{3}{4}\cdot\frac{1}{6}x^6 + \frac{5}{3}\cdot\frac{1}{3}x^3 - \frac{1}{2}\cdot\frac{1}{2}x^2 + C$

$= \frac{1}{8}x^6 + \frac{5}{9}x^3 - \frac{1}{4}x^2 + C$

9) $\int \left(\pi x + \frac{1}{\pi}\right) dx$

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$= \frac{\pi}{2}x^2 + \frac{1}{\pi}x + C$

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$\frac{d}{dx}[x^n]=nx^{n-1}$ - the power rule	$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$ - the power rule reversed

10) $\int \frac{1}{x^2}\,dx$

$= \int x^{-2}\,dx$

$= \frac{x^{-1}}{-1} + C$

$= \frac{-1}{x} + C$

$= -x^{-1} + C$

11) $\int \left(\frac{4}{x^3} - \frac{5}{x^4}\right) dx$

$= \int \left(4x^{-3} - 5x^{-4}\right) dx$

$= \frac{4\,x^{-2}}{-2} - \frac{5\cdot x^{-3}}{-3} + C$

$= -2x^{-2} + \frac{5}{3}x^{-3} + C$

$= \frac{-2}{x^2} + \frac{5}{3x^3} + C$

12) $\int \sqrt{x}\,dx$

$$\int \sqrt{x}\, dx = \int x^{1/2}\, dx$$

$$= \frac{2}{3} x^{3/2} + C$$

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13) $\int \left(2\sqrt[3]{y} - 4\sqrt[4]{y}\right) dy$

14) $\int \left(\frac{1}{\sqrt{x}} - x^{2/3}\right) dx$

15) $\int \left(x^{\pi} + \sqrt{\pi}\right) dx$

13) $\int \left(2\sqrt[3]{y} - 4\sqrt[4]{y}\right) dy$

14) $\int \left(\frac{1}{\sqrt{x}} - x^{2/3}\right) dx$

15) $\int \left(x^{\pi} + \sqrt{\pi}\right) dx$

p 121

In taking integrals, you may have to be clever. There are only a certain set of rules and if an integration problem doesn't fit one of the rules, you may have to change the expression so that it does. I call this a "bag of tricks."

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Trick 1 - multiply out, then integrate

16) $\int (2x-3)^2\, dx = \int (4x^2 - 12x + 9)\, dx$

$= \frac{4}{3}x^3 - 6x^2 + 9x + C$

In taking integrals, you may have to be clever. There are only a certain set of rules and if an integration problem doesn't fit one of the rules, you may have to change the expression so that it does. I call this a "bag of tricks."

$$\int x^{-4}\left(x^2+3x+1\right)dx$$

Trick 2 - Split into individual fractions, then integrate

17) $\int \frac{x^2+3x+1}{x^4}\,dx =$ ______ 18) $\int \frac{(2x-5)(3x+2)}{\sqrt{x}}\,dx$

$$=\int\left(\frac{x^2}{x^4}+\frac{3x}{x^4}+\frac{1}{x^4}\right)\cancel{dx}$$

$$=\int x^{-2}+3x^{-3}+x^{-4}\,\cancel{dx}$$

$$=-x^{-1}-\frac{3}{2}x^{-2}-\frac{1}{3}x^{-3}+C$$

We took derivatives of trig functions earlier in the year. So, natually, we should be able to go backwards and take integrals involving trig functions.

Differentiation formula	Integration formula
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 \, dx = -\cot x + C$

$$\int \sin x \, dx = -\int -\sin x \, dx = -1 \cdot \cos x + C$$

Differentiation and integration using the sine and cosine functions occur in calculus all the time and students always get confused with signs. A good way to remember is to follow this chart

	sin		
	◊		
	cos		
	◊		When you differentiate, you follow the chart down.
Differentiation	– sin	Integration	
	◊		When you integrate, you follow the chart up.
	– cos		
	◊		
	sin		

$$\frac{d}{dx}(-\cos x) = \frac{d}{dx}(-1 \cdot \cos x)$$

$$= -1 \cdot -\sin x$$

$d \downarrow$ S C -S -C S $\uparrow \int$

Examples - find the integral of each of the following:

20) $\int 4\sin x\,dx$

$= 4 \cdot \int \sin x\,dx$

$= 4 \cdot -\cos x + C$

$= -4\cos x + C$

21) $\int \frac{-2\cos x}{3}\,dx$

$= \frac{-2}{3}\int \cos x\,dx$

$= \frac{-2}{3}\sin x + C$

22) $\int \frac{5}{\cos^2 x}\,dx$

$= 5\int \frac{1}{\cos^2 x}\,dx$

$= 5\int \left(\frac{1}{\cos x}\right)^2 dx$

$= 5\int (\sec x)^2\,dx$

$= \int 5\sec^2 x\,dx$

$= 5\tan x + C$

23) $\int (4\cos x - 9\sin x)\, dx$

$4\sin x - 9 \cdot -\cos x + C$

$4\sin x + 9\cos x + C$

24) $\int \left(\frac{-\sin x}{\cos^2 x}\right) dx$

$\int -\sin x \cdot \frac{1}{\cos^2 x}\, dx$

$-\int \sin x \sec^2 x\, dx$

$-\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\, dx$

$-\int \sec x \tan x\, dx$

$-\sec x + C$

25) $\int (\theta^2 - 2\csc^2\theta)\, d\theta$

23) $\int (4\cos x - 9\sin x)\, dx$

24) $\int \left(\frac{-\sin x}{\cos^2 x}\right) dx$

25) $\int (\theta^2 - 2\csc^2\theta)\, d\theta$

$$\frac{\theta^3}{3} + 2\cot\theta + C$$

p 123

If you were given the statement that $\frac{dy}{dx}=4x$, we can cross multiply to get $dy=4x\,dx$. We can now integrate each side of the equation to get $\int dy=\int 4x\,dx$. From there, we can solve for y.

The original statement $\frac{dy}{dx}=4x$ is called a differential equation (DEQ). In a differential equation, you are given a statement about the derivative of y, $\frac{dy}{dx}$. Your goal is to solve for y. We have done so with the exception of the $+C$, the constant of integration. So we have a *general solution* of the DEQ. But suppose we were told that if $x = 0$, then $y = 5$. From there we can solve for C and we will thus have the *specific solution* of the DEQ. Let's do so.

Example 18) Solve the differential equation.

$f'(x)=3x-1,\ \boxed{f(2)=3}$

$f'(x)=3x-1$

$f(x)=\frac{3x^2}{2}-x+C$ ← general sol

$f(2)=\frac{3\cdot 4}{2}-2+C=3$

$C=-1$

$\boxed{f(x)=\frac{3x^2}{2}-x-1}$ ← particular solution

Example 19) Solve the differential equation.

$f'(x)=x^2-2x+2,\ f(3)=-1$

$f(x)=\frac{x^3}{3}-x^2+2x+C$

$f(3)=9-9+6+C=-1$

$C=-7$

$\boxed{f(x)=\frac{x^3}{3}-x^2+2x-7}$

Example 20) Solve the differential equation.

$$f''(x)=2,\ f'(4)=1,\ f(-1)=2$$

$f'(x)=2x+C$

$f'(4)=8+C=1$

$C=-7$

$f'(x)=2x-7$

$f(x)=x^2-7x+C$

$f(-1)=1+7+C=2$

$C=-6$

$f(x)=x^2-7x-6$

Example 21) Solve the differential equation.

$$f''(x)=2x,\ f'(-5)=30,\ f(2)=-1$$

$f'(x)=x^2+C$

$f'(-5)=25+C=30$

$C=5$

$f'(x)=x^2+5$

$f(x)=\frac{x^3}{3}+5x+C$

$f(2)=\frac{8}{3}+10+C=-1$

$C=-\frac{3}{3}-\frac{38}{3}$

$f(x)=\frac{x^3}{3}+5x-\frac{41}{3}$

Example 22) Given that the graph of $f(x)$ passes through the point (1, 6) and that the slope of its tangent line at $(x, f(x))$ is $\mathbf{2x+1}$, find $\boxed{f(6)}$.

$f'(x) = 2x+1 \quad ; \quad f(1) = 6$

$f(x) = $ ____________

$f(6) = \boxed{46}$