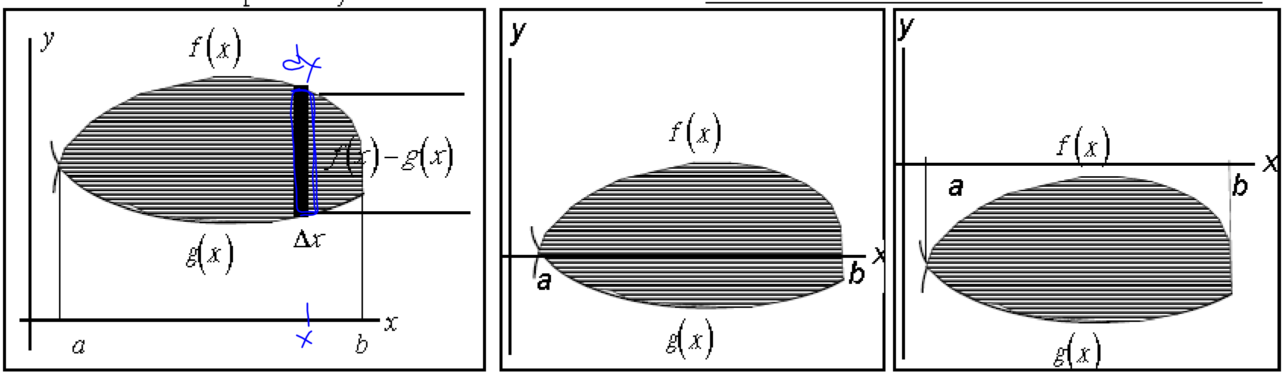
p167

Area of a Region Between Two Curves - Classwork

Consider two functions f and g that are continuous on $[a,b]$. If the graphs of both functions are above the x-axis and $f(x)\geq g(x)$, we can find the area between the two graphs as the area of the region under the graph of g subtracted from the area of f. Picture below left. The area of a represented rectangle is width times height. The base is Δx and the height of the rectangle is $f(x)-g(x)$. So the area of the rectangle is $(f(x)-g(x))\,\Delta x$. As Δx goes to zero, we have a Reimann sum - $\lim_{n\to\infty}\left[\sum_{i=1}^{n}(f(x_i)-g(x_i))\,\Delta x\right]$ which we know to be $A=\int_a^b [f(x)-g(x)]\,dx$.

So we say that the area of the shaded region is $A=\int_a^b [f(x)-g(x)]\,dx$. Now suppose the curves are not both above the x-axis? Explain why that doesn't make a difference. ______________________________



$A=\int_a^b [f(x)-g(x)]\,dx$ $A=\int_a^b [f(x)-g(x)]\,dx$ $A=\int_a^b [f(x)-g(x)]\,dx$

$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

As we go through problems in this section, we will use extensively the calculator's ability to find numerical integrals. We will not go through the FTC - integrating and plugging in the top and bottom limits. On the A.P. test, you are allowed to use the calculator's ability to numerically integrate as long as you show the proper setup.

Example 1) Find the region bounded by the graphs of $y = x^2 + x + 2$, $y = -x$, $x = 0$ and $x = 1$.

First sketch it

The integral is ______________

The area is $10/3$

$$\int_0^1 \left(x^2 + x + 2 + + x\right) dx$$

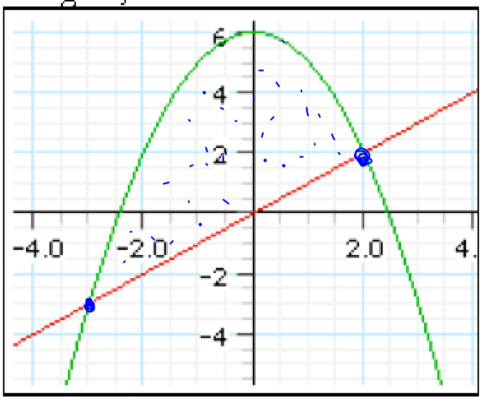
$$= \frac{x^3}{3} + x^2 + 2 \Big|_0^1$$

$$= \frac{1}{3} + 1 + 2$$

$$= 3\tfrac{1}{3}$$

$$= 10/3$$

Example 2) Find the region bounded by the graphs $y = 6 - x^2$ and $y = x$. Sketch it.
To integrate, we must find intersections.



The integral is ______________________

The area is ______________________

$6 - x^2 = x$

$0 = x^2 + x - 6$

$0 = (x+3)(x-2)$

$x = -3 \quad x = 2$

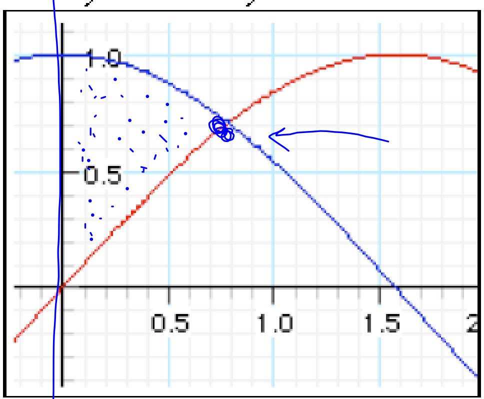
$= \int_{-3}^{2} (6 - x^2 - x)\,dx$

$= \left(6x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_{-3}^{2}$

$= 12 - \frac{8}{3} - 2 - \left(-18 + 9 - \frac{9}{2}\right)$

$= \frac{125}{6}$

Example 3) Find the region bounded by the graphs $x = 0$, and the first intersection of $y = \sin x$ and $y = \cos x$.



sin x

cos x

Your first job is to find the intersection of these two curves. Do it algebraically.

The integral is ______________________

The area is __________________________

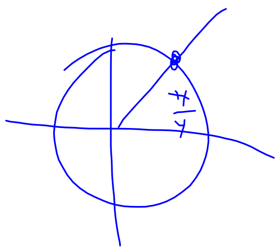
$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0 + 1)$$

$$= \boxed{\sqrt{2} - 1}$$

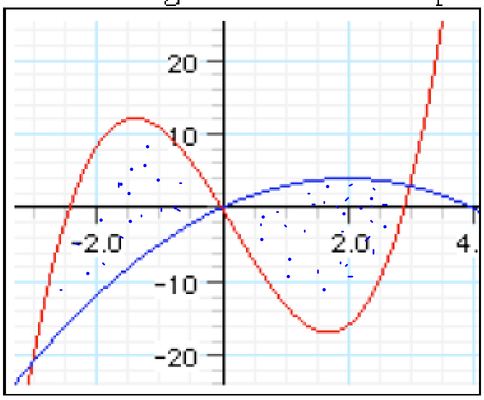
$\sin x = \cos x$



Example 4) Find the region bounded by

$y_1 = 2x^3 - x^2 - 14x$ and $y_2 = 4x - x^2$.

Sketch it noting which curve is on top.



Find the intesections.

The integral is ______________

The area is ______________

$2x^3 - x^2 - 14x = 4x - x^2$

$2x^3 - 18x = 0$

$2x(x^2 - 9) = 0$

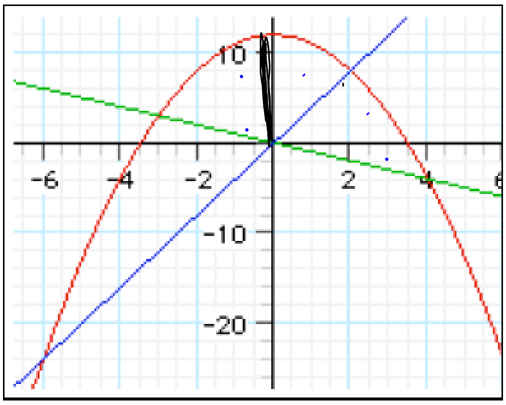
$x = 0 \quad x = \pm 3$

$\int_{-3}^{0} (y_1 - y_2)\,dx + \int_{0}^{3} (y_2 - y_1)\,dx$

OR

$$\boxed{\int_{-3}^{0} \left(2x^3 - x^2 - 14x - (4x - x^2)\right) dx + \int_{0}^{3} \left(4x - x^2 - 2x^3 + x^2 + 14x\right) dx}$$

Example 5) Find the region bounded by the graphs $y = 12 - x^2$, $y = -x$ and $y = 4x$.



Shade the region you are trying to find.

The integral is ____________________

The area is ______________________

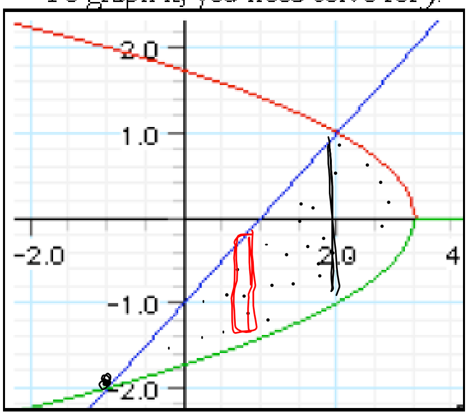
$$12 - x^2 = 4x$$
$$0 = x^2 + 4x - 12$$
$$= (x+6)(x-2)$$

$$= \int_{-6}^{0} (12 - x^2 - 4x)\,dx + \int_{0}^{4} (12 - x^2 + x)\,dx$$

Example 6) Find the region bounded by

$x = 3 - y^2$ and $x = y + 1$.

To graph it, you need solve for y.



$x = 3 - y^2$

$y^2 = 3 - x$

$y = \pm\sqrt{3 - x}$

$y = x - 1$

$$\int_{-1}^{2} \left((x-1) - -\sqrt{3-x}\right) dx$$

$$+ \int_{2}^{3} \left(\sqrt{3-x} - -\sqrt{3-x}\right) dx$$

Rather than integrating using x, use y
and go right to left instead of top to bottom

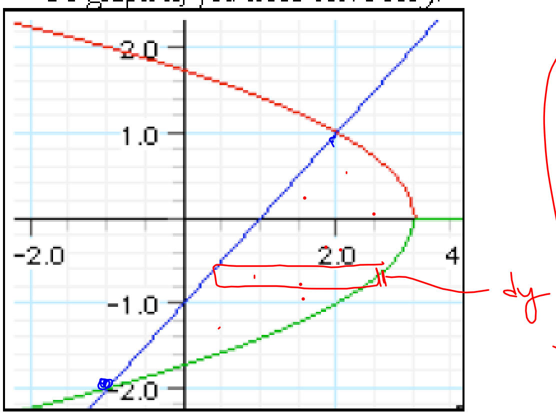
The integral is ____________________

The area is ______________________

Example 6) Find the region bounded by

$x = 3 - y^2$ and $x = y + 1$.

To graph it, you need solve for y.



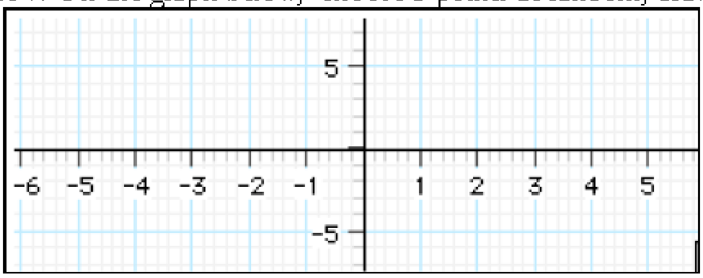
$$\int_{-2}^{1} \left(3 - y^2 - (y+1) \right) dy$$

Rather than integrating using x, use y and go right to left instead of top to bottom

The integral is ____________________

The area is 4.5

Example 7. On the graph below, choose 3 points at random, draw the triangle, and find its area.



Area of a Region Between Two Curves - Homework

For each problem, sketch the region bounded by the graphs of the functions and find the region of the area.

1. $y = x^2 + 2x + 1, y = 2x + 5$

2. $y = x^2 - 4x + 3,$
   $y = -x^2 + 2x + 3$

3. $y = x^2$
   $y = x^3$

Integral: ________________
Area: ___________________

Integral: ________________
Area: ___________________

Integral: ________________
Area: ___________________

4. $y=(x-1)^3, y=x-1$

Integral: ____________________

Area:________________________

5. $y=\frac{1}{x^2}, y=0, x=1, x=5$

Integral: ____________________

Area:________________________

6. $y=\sqrt{3x}+1, y=x, x=0$

Integral: ____________________

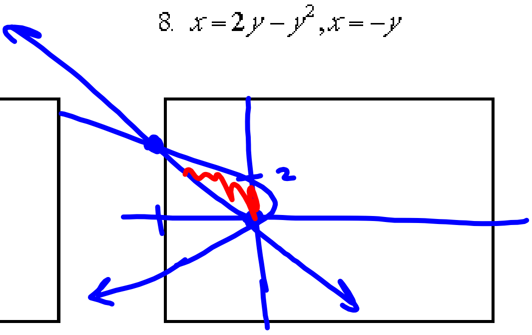
Area:________________________

7. $y=\sqrt[3]{x}, y=x$

Integral: ________________

Area: ____________________

8. $x=2y-y^2, x=-y$



Integral: ________________

Area: ____________________

9. $y=\frac{1}{1+x^2}, y=\frac{1}{2}x^2$

Integral: ________________

Area: ____________________

$$\int_0^3 \left(2y-y^2 - -y\right)dy$$

10. $y = 2\sin x, y = \tan x$
$-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

Integral: ____________________
Area: ________________________

11. $y = 2\sin x, y = \cos 2x$
$0 \le x \le \pi$

Integral: ____________________
Area: ________________________

12. $y = x^3$ and the tangent to y at $(1,1)$

Integral: ____________________
Area: ________________________

HW

do p166 #5-7

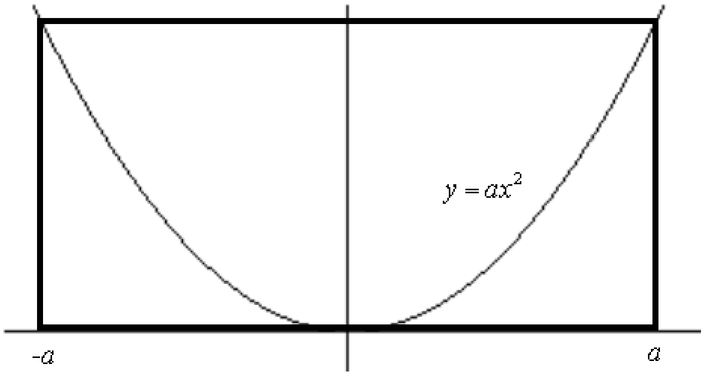
do p 169-170 (odds)

(For now... to make sure you're on the right track)

13. Find the value(s) of b if the vertical line $x = b$ divides the region between $y = \mathbf{16} - \mathbf{2}x$ and the x and y-axis into 2 equal areas.

14. Use integration to to find the area of the triangle having the vertices (3, -2), (5, 7), and (7, 2)

15. Show that the area under the function $y = ax^2$ is $\frac{1}{3}$ of the area of the circumscribed rectangle.



$\cancel{2x}+5 = x^2 + \cancel{2x} + 1$

$0 = x^2 - 4$

$x^2 = x^3$

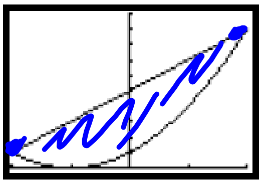
$0 = x^3 - x^2$

$= x^2(x-1)$

Area of a Region Between Two Curves - Homework

For each problem, sketch the region bounded by the graphs of the functions and find the region of the area.

1. $y_1 = x^2 + 2x + 1, y_2 = 2x + 5$



$\int_{-2}^{2} \left[\cancel{2x} + 5 - (x^2 + \cancel{2x} + 1)\right] dx$

Area $= 10.667$

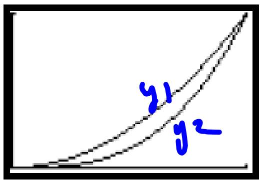
$$\int_{-2}^{2} (y_2 - y_1)\, dx$$

$$\int_{-2}^{2} (-x^2 + 4)\, dx = -\frac{x^3}{3} + 4x \Big|_{-2}^{2}$$

$$= -\frac{8}{3} + 8 - \left(\frac{8}{3} - 8\right)$$

3. $y_1 = x^2$

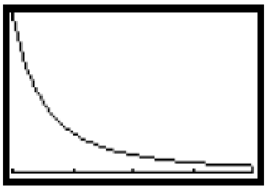
$y_2 = x^3$



$\int_{0}^{1} \left[x^2 - x^3\right] dx$

Area $= \frac{1}{12}$

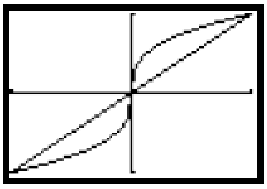
5. $y=\frac{1}{x^2}, y=0, x=1, x=5$



$\int_1^5\left[\frac{1}{x^2}\right]dx$

$\text{Area } = \frac{4}{5}$

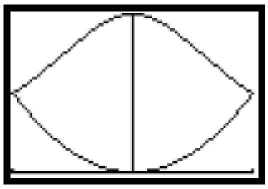
7. $y=\sqrt[3]{x}, y=x$



$2\int_0^1\left[\sqrt[3]{x}-x\right]dx$

$\text{Area } = \frac{1}{2}$

9. $y=\frac{1}{1+x^2}, y=\frac{1}{2}x^2$

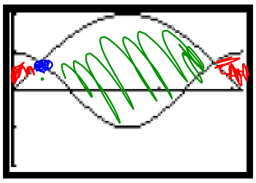


$2\int_0^1\left[\frac{1}{1+x^2}-\frac{1}{2}x^2\right]dx$

$\text{Area } = 1.237$

11. $y_1 = 2\sin x,\ y_2 = \cos 2x$

$0 \le x \le \pi$

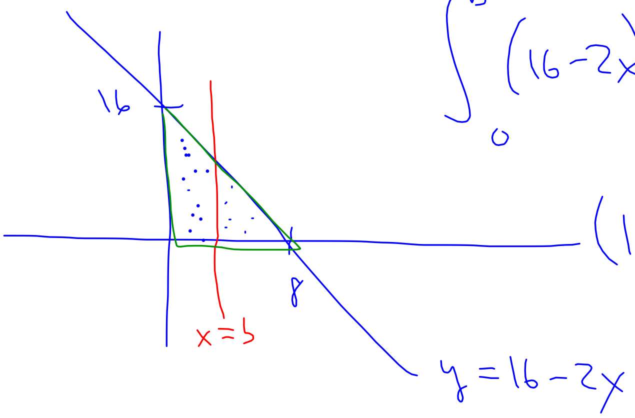


$$2\int_0^{.375}[\cos 2x - 2\sin x]\,dx + \int_{.375}^{2.767}[2\sin x - \cos 2x]\,dx$$

Area $= 4.807$

$$2\int_0^A (y_2 - y_1)\,dx + \int_A^B (y_1 - y_2)\,dx$$

13. Find the value(s) of b if the vertical line $x = b$ divides the region between $y = 16 - 2x$ and the x and y-axis



$$\int_0^b (16-2x)\,dx = \int_b^8 (16-2x)\,dx$$

$$(16x - x^2)\Big|_0^b = (16x - x^2)\Big|_b^8$$

$$16b - b^2 = 128 - 64 - (16b - b^2)$$

14.

$$\frac{1}{2}\int_0^8 (16-2x)\,dx = \int_0^b (16-2x)\,dx$$

$$0 = 2b^2 - 32b + 64$$

$$0 = 2(b^2 - 16b + 32)$$

$$b \approx 2.343$$

15. Show that the area under the function $y = ax^2$ is $\frac{1}{3}$ of the area of the circumscribed rectangle.

$$A = \int_0^a ax^2\,dx = \frac{2}{3}\left[ax^3\right]_0^a = \frac{2}{3}a^4$$

$$A_{\text{rect}} = 2a\left(a^3\right) = 2a^4$$