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Differentiation of the Natural Log Function - Classwork

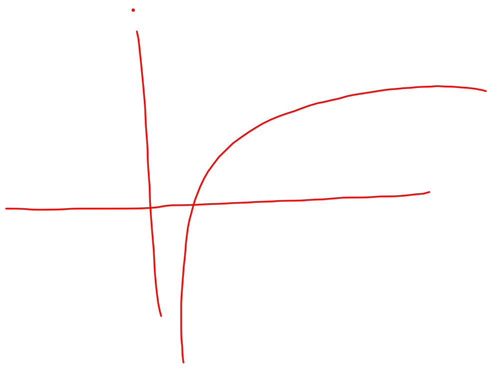
We have examined derivatives using the power rule, product rule, quotient rule, and trig. But what about the derivative of $y = \ln x$? We have no rule to cover such functions. Let's see how your calculator deals with it. In your TI-83 calculator, let Y1 = nDerviv(ln*x*,*x*,*x*) and set up your table with x starting at 1 and $\Delta x = 1$.

x	0	1	2	3	4	5	6	7	8	9	10
y'	ERROR	1.000	0.500	0.333	0.250	0.200	0.167	0.143	0.125	0.111	0.100

As you look at the relationship between the value of x and the vealue of the derivative of $\ln x$, it should be clear. What is it? the derivative is just the reciprocal of x

$y = \ln x$

$y'(4) = \frac{1}{4}$



So we have a new differentiation rule - the ln rule:

$$\frac{d}{dx}[\ln x]=\frac{1}{x}, x>0 \qquad \frac{d}{dx}[\ln u]=\frac{1}{u}\cdot\frac{du}{dx}=\frac{u'}{u}, u>0 \qquad \frac{d}{dx}[\ln|u|]=\frac{1}{u}\cdot\frac{du}{dx}=\frac{u'}{u}, u\neq 0$$

What this says is to take the derivative of the ln of some expressions, you simply use the reciprocal of the expression multipled by the derivative of that expression. The expression must be a positive number.

Examples) Find the derivative of the following expressions:

$M = \ln x$
$M' = 1/x$

$B = \sqrt{x}$
$B' = \frac{1}{2\sqrt{x}}$

1) $y=\ln(4x)$

$$y' = \frac{1}{4x}\cdot 4$$

$$y' = \frac{1}{x}$$

2) $y=\ln(x^2-3)$

$$y' = \frac{2x}{x^2-3}$$

3) $y=\ln(3x^2-5x+8)$

$$y' = \frac{6x-5}{3x^2-5x+8}$$

$B = x^{1/2}$

4) $y=\ln\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}$$

$$y' = \frac{1}{2x}$$

$$y' = \frac{1}{x^{1/2}}\cdot\frac{1}{2}x^{-1/2}$$

$$= \frac{1}{2}\cdot\frac{1}{x}$$

$$= \frac{1}{2x}$$

$(\ln x)^2 = \ln^2 x$

You now have 5 sets of rules - power, product, quotient, trig (6 rules), and now "ln". Just because there is an "ln" in the problem does not mean it uses the ln rule above.

Examples) Find the derivative of the following expressions:

5) $y = x^2 \ln x$

P. Rule

$y' = x^2 \cdot \frac{1}{x} + \ln x \cdot 2x$

$= x + \ln x \cdot 2x$

$= x + 2x \ln x$

6) $y = \frac{\ln x}{x}$

Q rule

$y' = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$

$= \frac{1 - \ln x}{x^2}$

7) $y = \frac{x}{\ln x}$

Q rule

$y' = \frac{\ln x \cdot 1 - x \cdot \frac{1}{x}}{\ln^2 x}$

$= \frac{\ln x - 1}{\ln^2 x}$

8) $y = (\ln x)^5$

CR.

$y' = 5(\ln x)^4 \cdot \frac{1}{x}$

$y' = \frac{5(\ln x)^4}{x}$

9) $y = \sqrt{\ln x} = (\ln x)^{1/2}$

CR

$y' = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{1}{x}$

$y' = \frac{1}{2x\sqrt{\ln x}}$

~~cos ln x~~

10) $y = \cos(\ln x)$

CR

$y' = -\sin(\ln x) \cdot \frac{1}{x}$

$y' = \frac{-\sin(\ln x)}{x}$

~~ln · cos x~~

11) $y = \ln(\cos x)$

CR

$y' = \frac{1}{\cos x} \cdot -\sin x$

$y' = \frac{-\sin x}{\cos x}$

$= -\tan x$

$M = \ln x \quad B = \ln x$

$M' = \frac{1}{x} \quad B' = \frac{1}{x}$

12) $y = \ln(\ln x)$

CR

$y' = \frac{1}{\ln x} \cdot \frac{1}{x}$

$= \frac{1}{x \ln x}$

9) $y = \sqrt{\ln x} = (\ln x)^{1/2}$

CR

$y' = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{1}{x}$

$y' = \frac{1}{2x\sqrt{\ln x}}$

10) $y = \cos(\ln x)$

$\cancel{\cos \cdot \ln x}$

CR

$y' = -\sin(\ln x) \cdot \frac{1}{x}$

$y' = \frac{-\sin(\ln x)}{x}$

11) $y = \ln(\cos x)$

$\cancel{\ln \cdot \cos x}$

CR

$y' = \frac{1}{\cos x} \cdot -\sin x$

$y' = \frac{-\sin x}{\cos x}$

$= -\tan x$

12) $y = \ln(\ln x)$

$M = \ln x \quad B = \ln x$

$M' = \frac{1}{x} \quad B' = \frac{1}{x}$

CR

$y' = \frac{1}{\ln x} \cdot \frac{1}{x}$

$= \frac{1}{x \ln x}$

Remember your log rules. They can help you to take derivative of harder expressions.

1. $\log(a\cdot b)=\log a+\log b$	2. $\log\left(\frac{a}{b}\right)=\log a-\log b$	3. $\log a^b=b\log a$

Examples) Find the derivative dy/dx of the following expressions:

13) $y=\ln\sqrt{x^2+2x-3}$

Hard way: Use CR

$$y'=\frac{1}{\sqrt{x^2+2x-3}}\cdot\frac{1}{2}(x^2+2x-3)^{-1/2}\cdot(2x+2)$$

$$=\frac{1}{\sqrt{x^2+2x-3}}\cdot\frac{1}{\cancel{2}\sqrt{x^2+2x-3}}\cdot\cancel{2}(x+1)$$

$$=\frac{x+1}{x^2+2x-3}$$

Easy way: Use log rules

$$y=\ln(x^2+2x-3)^{1/2}$$

$$=\frac{1}{2}\ln(x^2+2x-3)$$

$$y'=\frac{1}{2}\cdot\frac{2x+2}{x^2+2x-3}$$

$$y'=\frac{x+1}{x^2+2x-3}$$

$$y=\frac{1}{2}\ln\left[(x+3)(x-1)\right]$$

$$=\frac{1}{2}\left(\ln(x+3)+\ln(x-1)\right)$$

$$y'=\frac{1}{2}\left(\frac{1}{x+3}+\frac{1}{x-1}\right)$$

14) $y = \ln \frac{x^2}{3x-2} = \ln x^2 - \ln(3x-2)$

$= 2\ln x - \ln(3x-2)$

$y' = \frac{2}{x} - \frac{3}{3x-2}$

$y' = \frac{1}{\frac{x^2}{3x-2}} \cdot \frac{(3x-2)(2x) - x^2 \cdot 3}{(3x-2)^2}$

$= \frac{3x-2}{x^2} \cdot \frac{6x^2 - 4x - 3x^2}{(3x-2)^2}$

$= \frac{3x^2 - 4x}{x^2(3x-2)}$

$= \frac{3x-4}{x(3x-2)}$

15) $y = \ln\left(x\sqrt{3x-1}\right)$

$$y = \ln x + \frac{1}{2}\ln(3x-1)$$

$$y' = \frac{1}{x} + \frac{1}{2} \cdot \frac{3}{3x-1}$$

$$= \frac{1}{x} + \frac{3}{2(3x-1)}$$

16) $y = \ln\sqrt{\dfrac{x^2+1}{x^2-1}}$

$$y = \frac{1}{2}\left(\ln(x^2+1) - \ln(x^2-1)\right)$$

$$y' = \frac{1}{2}\cdot\left(\frac{2x}{x^2+1} - \frac{x^2}{x^2-1}\right)$$

17) $y = \ln \dfrac{x(x^2+3)^3}{\sqrt[3]{2x^2+4}}$

$$y = \ln x + 3\ln(x^2+3) - \frac{1}{3}\ln(2x^2+4)$$

$$y' = \frac{1}{x} + \frac{6x}{x^2+3} - \frac{4x}{3(2x^2+4)}$$

18) $x^2 - 3\ln y + y^2 = 25$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(2y - \frac{3}{y}\right) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y - 3/y}$$

19) Find relative extrema of

$y = \ln\left(x^2 + 4x + 6\right)$

$$y' = \frac{2x + 4}{x^2 + 4x + 6} \stackrel{?}{=} 0$$

$x = -2$

rel min

y' $-$ | $+$

-2

y has a rel. min at $x = -2$ of $y(-2) = \ln 2$.

20) Find the equation of the tangent line to $y = 4x^2 - \ln x$ at $(1,4)$

$$y' = 8x - \frac{1}{x}$$

$$y'(1) = 8 - 1 = 7$$

$$\boxed{y - 4 = 7(x - 1)}$$

A technique not included on the AP exam but helpful to taking hard derivatives is called logarithmic differentiation. It essentially says to take the natural log of the hard expression in order to take advantage of the log rules. Example 20) $y = \frac{(2x-1)^3}{\sqrt{x^2+x+1}}$

$$\ln y = \ln\left(\quad \right)$$

$$\ln y = 3\ln(2x-1) - \frac{1}{2}\ln(x^2+x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{6}{2x-1} - \frac{2x+1}{2(x^2+x+1)}$$

$$\frac{dy}{dx} = \left(\frac{6}{2x-1} - \frac{2x+1}{2(x^2+x+1)}\right)\left(\frac{(2x-1)^3}{\sqrt{x^2+x+1}}\right)$$