p195

Derivatives and Integrals of Expressions with "e" - Classwork

Let us try to take the derivative of $y = e^x$. Again, it seems as if that there is no rule (power, product, quotient, trig, ln) to take it. Let's examine this by use of the calculator. Set Y1 = e^x and Y2 = NDeriv(e^x,x,x). Then set a table to look at these values.

x	-4	-3	-2	-1	0	1	2	3	4
e^x	0.018	0.050	0.135	0.368	1.000	2.718	7.389	20.086	54.598
$\frac{d}{dx}\left(e^x\right)$	0.018	0.050	0.135	0.368	1.000	2.718	7.389	20.086	54.598

It should be obvious (and surprising) what is happening. Let's try and prove it. Let's take the derivative of $y = e^x$ using logarithmic differentiation.

So we end up with the result that equations in the form of $y = Ce^x$ are the only equations (with the exception of $y = 0$) whose derivative is the same as the expression itself.

$$\boxed{\frac{d}{dx}\left[e^x\right] = e^x \text{ and if } u \text{ is a differentiable function of } x \text{ then } \frac{d}{dx}\left[e^u\right] = e^u \frac{du}{dx}}$$

Examples) Find the derivative dy/dx of the following expressions:

1) $y = e^{5x}$

$y' = e^{5x} \cdot 5$

$= 5e^{5x}$

2) $y = 4e^{1-2x}$

$y' = 4 \cdot e^{1-2x} \cdot -2$

$y' = -8e^{1-2x}$

3) $y = e^{x^2-3x-1}$

$y' = (2x-3)e^{x^2-3x-1}$

4) $y = 2e^{\sqrt{x}}$

$y' = \frac{2e^{\sqrt{x}}}{2\sqrt{x}}$

$= \frac{e^{\sqrt{x}}}{\sqrt{x}}$

We now have 6 basic rules for derivatives: power, product, quotient, trig (6), ln, and now e.

5) $y=\left(e^x+3\right)^2$

CR

$y' = 2(e^x+3)\cdot e^x$

$= 2e^x(e^x+3)$

$= 2e^{2x} + 6e^x$

6) $y = xe^x$

PR

$y' = xe^x + e^x$

$y' = e^x(x+1)$

7) $y=\sin\left(e^x\right)$

CR

$y' = \cos(e^x)\cdot e^x$

$y' = e^x\cos(e^x)$

8) $y = e^{\sin x}$

CR

$y' = e^{\sin x}\cdot\cos x$

$y' = \cos x\cdot e^{\sin x}$

9) $y=\ln\left(x+e^x\right)$

$y' = \dfrac{1+e^x}{x+e^x}$

10) $y=\dfrac{e^{4x}}{x}$

$y' = \dfrac{x\cdot 4e^{4x} - e^{4x}}{x^2}$

$= \dfrac{e^{4x}(4x-1)}{x^2}$

11) $y=\dfrac{x}{e^{4x}}$

$y' = \dfrac{e^{4x} - x\cdot 4e^{4x}}{e^{8x}}$

$= \dfrac{e^{4x}(1-4x)}{e^{8x}}$

$= \dfrac{1-4x}{e^{4x}}$

12) $y=\sqrt[3]{e^x}$

$y = e^{x/3}$

$y' = \frac{1}{3}e^{x/3}$

Find dy/dx by implicit differentiation:

13) $xe^y + 8x - 3y = 0$

$$xe^y \frac{dy}{dx} + e^y + 8 - 3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(xe^y - 3\right) = -e^y - 8$$

$$\frac{dy}{dx} = \frac{-e^y - 8}{xe^y - 3}$$

Find the second derivative of the function:

14) $y = e^x - e^{-x}$

$y' = e^x + e^{-x}$

$y'' = e^x - e^{-x}$

15. Find relative extrema and inflection point(s) for the function $y = e^x - e^{-x}$.

$y' = e^x + e^{-x} = 0$

$e^x + \frac{1}{e^x} = 0$

$e^x \cdot e^x = \frac{-1}{e^x} \cdot e^x$

$e^{2x} = -1$

no soln, so extrema

$y' = e^x + e^{-x}$

$y'' = e^x - e^{-x} = 0$

$e^x = \frac{1}{e^x}$

$e^{2x} = 1$

$x = 0$

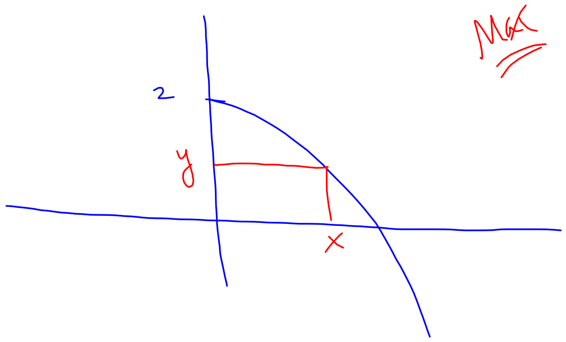
I.P.

y'' $-$ | $+$

$x=0$

$\boxed{\text{I.P. at } x=0}$

16) Find the area of the largest rectangle that can be inscribed under the curve $y = 3 - e^x$ in the first quadrant.



Max

Area = xy

Area $= x(3 - e^x) = 3x - xe^x$

Area' $= 3 - (xe^x + e^x) = 0$

$3 = e^x(x+1)$

(technology) $x \approx .618$

Area = .70747

Obviously, since the derivative of $y = e^x$ is e^x, it follows that the integral formula should be as simple.

$$\boxed{\int e^x dx = e^x + C \text{ and if } u \text{ is a differentiable function of } x \text{ then } \int e^u du = e^u + C}$$

Examples) Find the following:

17) $\int e^{5x} dx$

$u = 5x$

$du = 5dx$

$= \frac{1}{5}\int e^u du$

$= \frac{1}{5}e^{5x} + C$

18) $\int 4e^{1-x} dx$

$= -4e^{1-x} + C$

19) $\int 2xe^{x^2} dx$

$u = x^2$

$du = 2x\,dx$

$= \int e^u du$

$= e^u + C$

$= e^{x^2} + C$

20) $\int \frac{e^{1/x}}{2x^2} dx$

$u = \frac{1}{x}$

$du = -\frac{1}{x^2}$

$= -\frac{1}{2}\int e^u du$

$= -\frac{1}{2}e^u + C$

$= -\frac{1}{2}e^{1/x} + C$

21) $\int \cos x \cdot e^{\sin x}\,dx$

$u = \sin x$

$du = \cos x\,dx$

$= \int e^u\,du$

$= e^u + C$

$= e^{\sin x} + C$

22) $\int \frac{e^x}{4-e^x}\,dx$

$u = 4 - e^x$

$du = -e^x$

$= -\int \frac{1}{u}\,du$

$= -\ln|4-e^x| + C$

23) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

$u = e^x + e^{-x}$

$du = e^x - e^{-x}$

$= \int \frac{1}{u}\,du$

$= \ln|u| + C$

$= \ln|e^x + e^{-x}| + C$

$= \ln(e^x + e^{-x}) + C$

24) $\int \frac{e^{4x} + 2e^x + 1}{e^x}\,dx$

$= \int (e^{3x} + 2 + e^{-x})\,dx$

$= \frac{1}{3}e^{3x} + 2x - e^{-x} + C$

$e^x \cdot \sin e^x \stackrel{?}{=} 0$

$e^x = \pi$

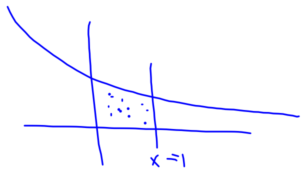
$x = \ln \pi$

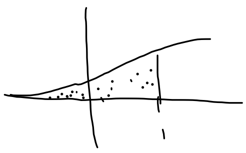
Find the area bounded by the curves and lines. Verify by calculator.

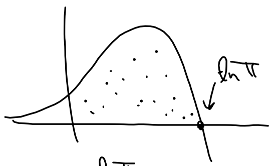
19) $y = e^{-x}, y = 0, x = 0, x = 1$

20) $y = \dfrac{e^x}{2 + e^x}, x = 1, y = 0$

21) $y = e^x \sin e^x, x = 0, y = 0$







$$A = \int_0^1 e^{-x}\,dx$$

$$= -e^{-x}\Big|_0^1$$

$$= -\left(e^{-1} - e^0\right)$$

$$= 1 - \frac{1}{e}$$

$$A = \int_{-\infty}^{1} \frac{e^x}{2+e^x}\,dx$$

$$= \ln\left(2+e^x\right)\Big|_{-\infty}^{1}$$

$$= \ln(2+e) - \ln(2)$$

$$= \ln\left(\frac{2+e}{2}\right)$$

$$\approx .858$$

$$A = \int_0^{\ln \pi} e^x \sin e^x\,dx$$

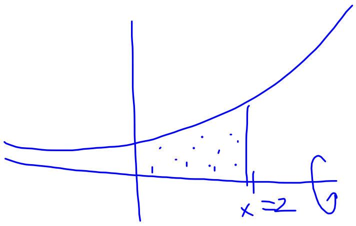
$$= -\cos e^x \Big|_0^{\ln \pi}$$

$$= -\left(\cos e^{\ln \pi} - \cos e^0\right)$$

$$= -\left(\cos \pi - \cos 1\right)$$

$$= 1 + \cos 1$$

22) Find the volume when the first quadrant region R bounded by $y = e^{x/2}$ and $x = 2$ is rotated about the x-axis.



$$V = \pi \int_0^2 \left(e^{x/2}\right)^2 dx$$

$$= \pi \cdot \int_0^2 e^x \, dx$$

$$= \pi \cdot e^x \Big|_0^2$$

$$= \pi \left(e^2 - 1\right)$$

Finally, occasionally, we have to take derivatives of exponential functions with bases other than e. Using the fact that $a^x = e^{(\ln a)x}$, we can take the derivative by saying that $\frac{d}{dx}a^x = e^{(\ln a)x} \cdot \ln a = a^x \cdot \ln a$. You need to know that:

$$\boxed{\frac{d}{dx}a^x = a^x \cdot \ln a \quad \text{and} \quad \frac{d}{dx}a^u = a^u \cdot \ln a \cdot \frac{du}{dx}}$$

Examples) Find the derivatives of:

23) $y = 5^x$

$y' = 5^x \cdot \ln 5$

24) $y = 3^{x^2-2x}$

$u = 3^x$

$u' = 3^x \ln 3$

24

$y' = 3^{x^2-2x} \cdot \ln 3 \cdot (2x-2)$

25) $y = x6^{-x}$

PR

25 $y' = x \cdot -6^{-x}\ln 6 + 6^{-x}$

$= 6^{-x}(1 - x\ln 6)$

Using logarithmic differentiation to find $\frac{d}{dx}[a^x]$

$y = a^x$

$\ln y = \ln a^x$

$\ln y = x \ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = \ln a \cdot y$

$\frac{dy}{dx} = \ln a \cdot a^x$