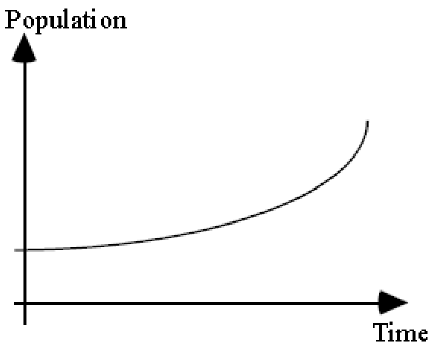
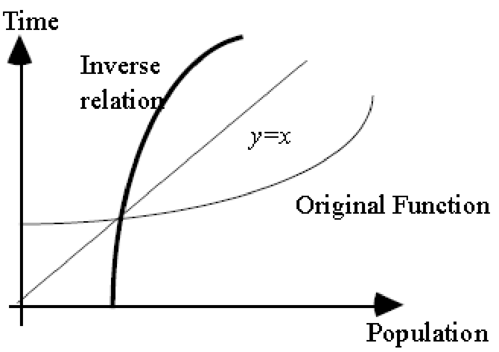
Inverse Trig Functions - Classwork

The left hand graph below shows how the population of a certain city may grow as a function of time. If you are interested in finding the time at which the population reaches a certain value, it may be more convenient to reverse the variables and write time as a function of population. The relation you get by interchanging the two variables is called the **inverse** of the original function. The graph of the inverse is shown on the right graph below.





For a linear function such as $y = 2x + 6$, interchanging the variables gives $x = 2y + 6$ for the inverse relation. Solving for y in terms of x gives $y = 0.5x - 3$. The symbol f^{-1}, pronounced "f inverse," is used for the inverse function. If $f(x) = 2x + 6$, then $f^{-1}(x) = 0.5x - 3$.

$y = x^5 - 1$

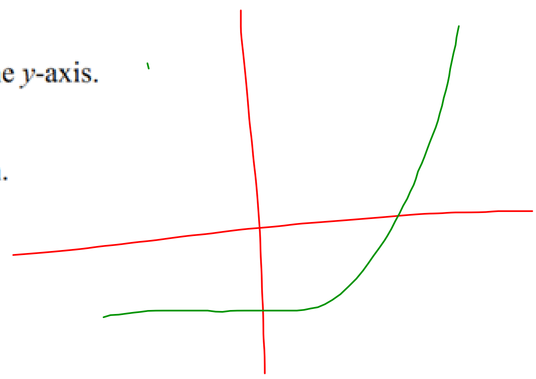
1. If the function f is defined by $f(x) = x^5 - 1$, then f^{-1}, the inverse function of f, is defined by $f^{-1}(x) =$

$x = y^5 - 1$

$y = \sqrt[5]{x+1}$

2. Suppose that f is a function that is defined for all real numbers. Which of the following conditions assures that f has an inverse function?

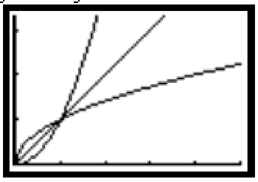
~~(A)~~ The function f is periodic.
~~(B)~~ The graph of f is symmetric with respect to the y-axis.
(C) The graph of f is concave up.
(D) The function f is a strictly increasing function.
~~(E)~~ The function f is continuous.



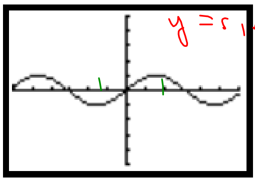
3. Let f and g be functions that are differentiable everywhere. If g is the inverse function of f and if $g(-2)=5$ and $f'(5)=-\frac{1}{2}$, then $g'(-2)=$

4. Find the derivative of the following : $f(x)=2\sin^{-1}\left(x^2-e^x\right)$

If f^{-1} turns out to be a function (passes the vertical line test), the the original function f is said to be **invertible**. Remember that the -1 exponent does *not* mean the reciprocal of $f(x)$. The inverse of a function undoes what the function did to x. That is $f^{-1}(f(x)) = x$. If $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$ and $\sqrt{x^2} = x$. Note that if the same scales are used for the two axes, then the graphs of f and f^{-1} are mirror images with respect to the 45° line $y = x$.



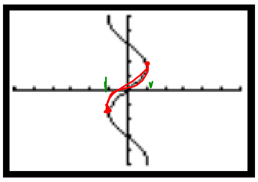
The inverses of the trigonometric functions follow from the definition. For instance, if the function is $y = \sin x$, the inverse function is given by $x = \sin y$. When we solve for y, we get $y = \sin^{-1} x$. The symbol **arcsin x** is sometimes used to help you distinguish $\sin^{-1} x$ from $\boxed{1/\sin x}$. Here are the graphs of $y = \sin x$ and $y = \sin^{-1} x$.



y = sin x

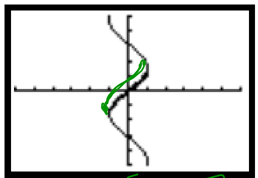
D: ℝ

R: [-1, 1]



D: [-1, 1]

R: ℝ



D: [-1, 1]

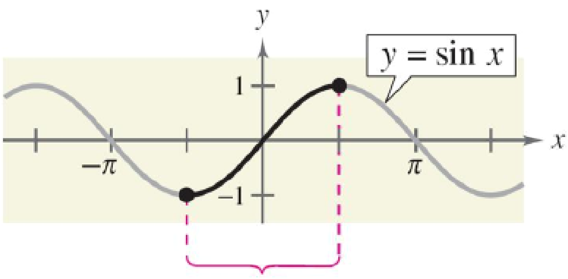
R: [-π/2, π/2]

4.7 INVERSE TRIGONOMETRIC FUNCTIONS

What You Should Learn

- Evaluate and graph the inverse sine function.
- Evaluate and graph the other inverse trigonometric functions.
- Evaluate and graph the compositions of trigonometric functions.

For a function to have an inverse function, it must be one-to-one—that is, it must pass the Horizontal Line Test.



sin x has an inverse function on this interval.

By definition, the values of inverse trigonometric functions are *always in radians*.

Restrict the domain to the interval $-\pi/2 \le x \le \pi/2$, the following properties hold.

1. On the interval $[-\pi/2, \pi/2]$, the function $y = \sin x$ is increasing.

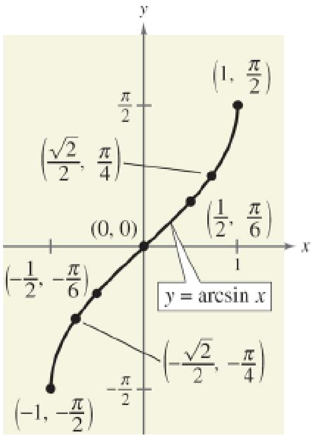
2. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range of values, $-1 \le \sin x \le 1$.

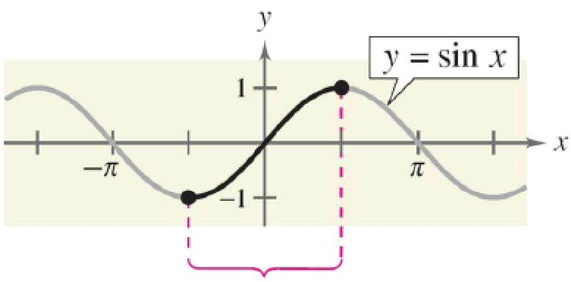
3. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ is one-to-one.

On the restricted domain $-\pi/2 \le x \le \pi/2$, $y = \sin x$ has a unique inverse function called the **inverse sine function.**

$$y = \arcsin x \quad \text{or} \quad y = \sin^{-1} x.$$

means the angle (or arc) whose sine is x.



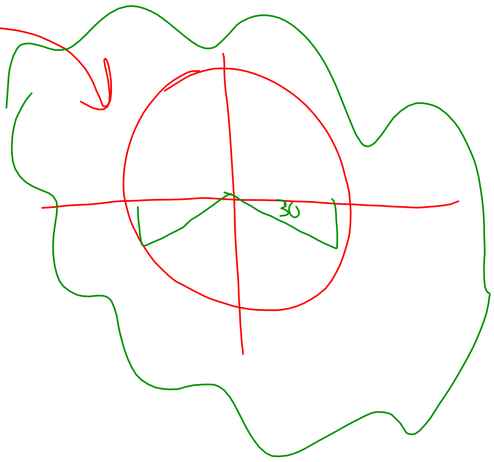


Example 1 – *Evaluating the Inverse Sine Function*

If possible, find the exact value.

a. $\arcsin\left(-\dfrac{1}{2}\right)$ **b.** $\sin^{-1}\dfrac{\sqrt{3}}{2}$ **c.** $\sin^{-1} 2$

$\sin\theta = -\frac{1}{2}$



a)

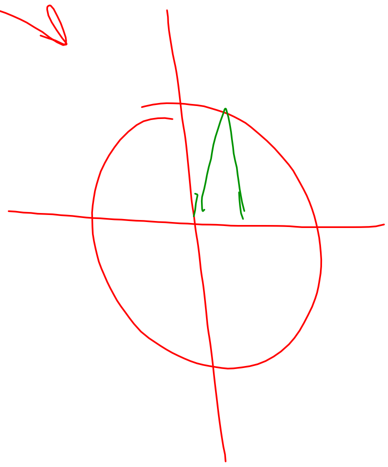
$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Example 1 – *Evaluating the Inverse Sine Function*

If possible, find the exact value.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\sin^{-1}\frac{\sqrt{3}}{2}$ **c.** $\sin^{-1}2$

$\sin\theta = \frac{\sqrt{3}}{2}$



$\sin^{-1}\frac{\sqrt{3}}{2} = \pi/3$

Example 1 – *Evaluating the Inverse Sine Function*

If possible, find the exact value.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\sin^{-1}\frac{\sqrt{3}}{2}$ **c.** $\sin^{-1} 2$

DNE

$\sin\theta = 2$

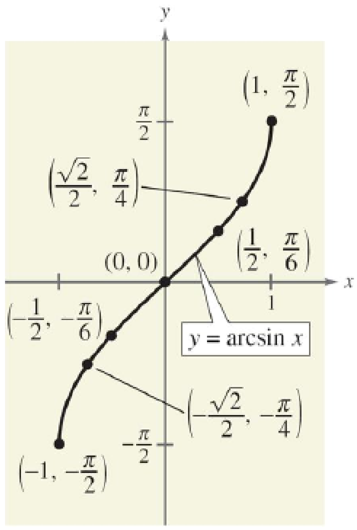
Example 2 – *Graphing the Arcsine Function*

Sketch a graph of $y = \arcsin x$.

Solution:

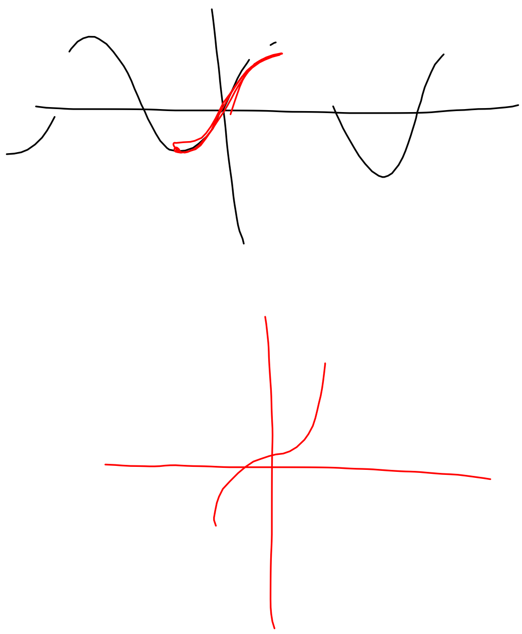
In the interval $[-\pi/2, \pi/2]$,

y	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x = \sin y$	-1	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1



Example 2 – *Solution*

$y = \arcsin x$

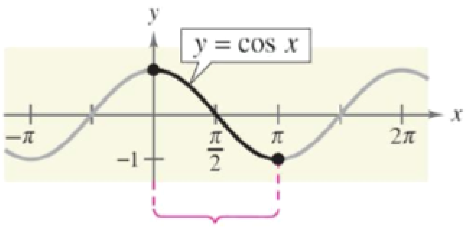


Domain: $[-1, 1]$

Range: $[-\pi/2, \pi/2]$

Other Inverse Trigonometric Functions

The cosine function is decreasing and one-to-one on the interval $0 \le x \le \pi$.



cos *x* has an inverse function on this interval.

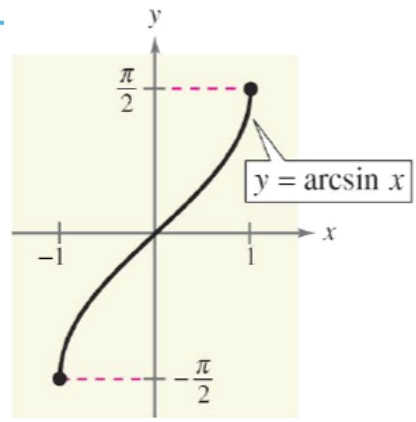
On the interval $0 \le x \le \pi$ the cosine function has an inverse function—the **inverse cosine function**:

$$y = \arccos x \quad \text{or} \quad y = \cos^{-1} x.$$

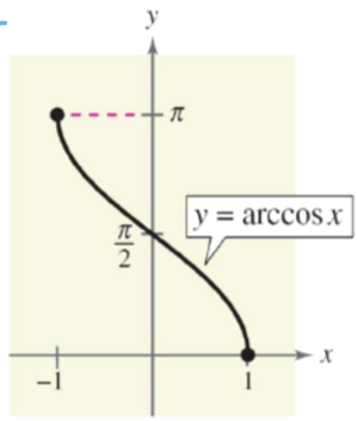
Other Inverse Trigonometric Functions

Definitions of the Inverse Trigonometric Functions

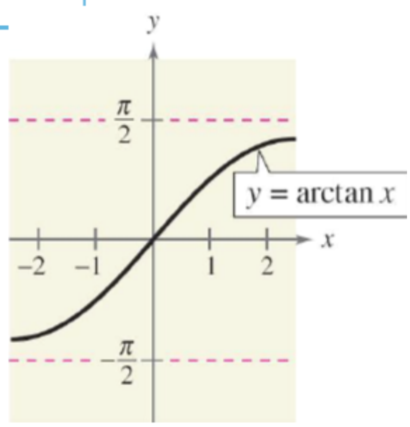
Function	*Domain*	*Range*
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$



DOMAIN: [–1,1]
RANGE: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



DOMAIN: [–1,1]
RANGE: [0, π]



DOMAIN: $(-\infty, \infty)$
RANGE: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

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Example 3 – *Evaluating Inverse Trigonometric Functions*

Find the exact value.

Check answers

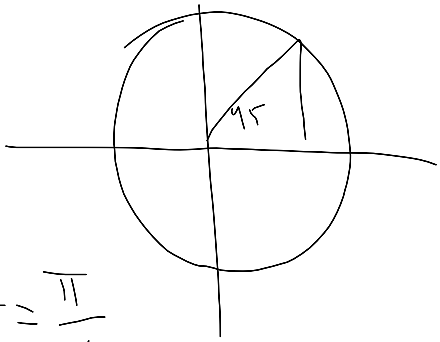
a. $\arccos \frac{\sqrt{2}}{2}$

b. $\cos^{-1}(-1)$ $= \pi$

c. $\arctan 0$ $= 0$

d. $\tan^{-1}(-1)$ $= -\frac{\pi}{4}$

a) $\cos\theta = \frac{\sqrt{2}}{2}$



a) $\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}$

Compositions of Functions

For all x in the domains of f and f^{-1}, inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Inverse Properties of Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \qquad \text{and} \qquad \arcsin(\sin y) = y.$$

If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\arccos x) = x \qquad \text{and} \qquad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \qquad \text{and} \qquad \arctan(\tan y) = y.$$

$f = x^2$

$g = \sqrt{x}$

$f(g(x)) = x$

$f(g(-2)) \stackrel{?}{=} -2$

Compositions of Functions

These inverse properties do not apply for arbitrary values of *x* and *y*.

$$\arcsin\left(\sin\frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2} \neq \frac{3\pi}{2}.$$

The property arcsin(sin *y*) = *y*

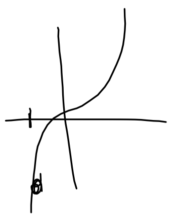
is not valid for values of *y* outside the interval $[-\pi/2, \pi/2]$.

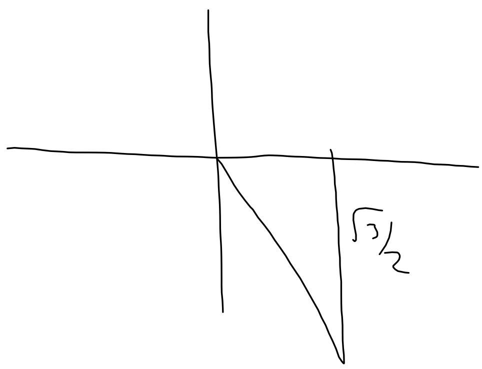
Example 5 – *Using Inverse Properties*

If possible, find the exact value.

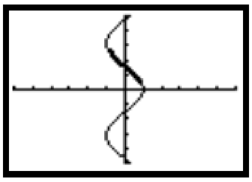
a. $\tan[\arctan(-5)]$ **b.** $\arcsin\left(\sin \frac{5\pi}{3}\right)$ **c.** $\cos(\cos^{-1}\pi)$ $= \pi$

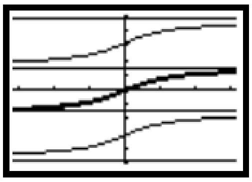
$= -5$ $= -\pi/3$





Below are pictures of the inverse cosine function and inverse tangent function. Note that in order to ensure that these relations are functions, we have to restrict the range.





So, we have these definitions:

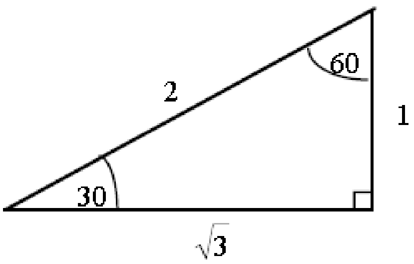
$$y = \sin^{-1} x \text{ if and only if } \sin y = x \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$y = \cos^{-1} x \text{ if and only if } \cos y = x \text{ and } y \in [0, \pi]$$
$$y = \tan^{-1} x \text{ if and only if } \tan y = x \text{ and } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

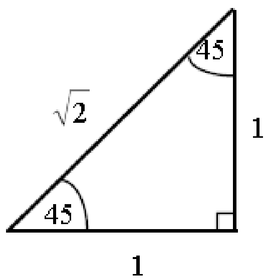
You must know conversions of degrees to radians and special triangles.

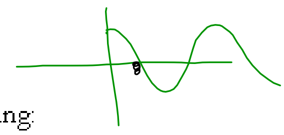
2π radians = 360° or π radians = 180°. Some of the relationships that you should know are:

Degrees	30	45	60	90	120	135	150	180
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

In a 30°-60°-90° triangle, the sides are always in the proportion $1 - \sqrt{3} - 2$.
In a 45°-45°-90° triangle, the sides are always in the proportion $1 - 1 - \sqrt{2}$.





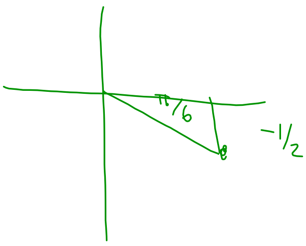


Example 1) Evaluate each of the following:

a) $\arcsin\left(-\frac{1}{2}\right) = \theta$

$= -\pi/6$

$-\frac{1}{2} = \sin\theta$



b. $\cos^{-1}0 = \theta$

$\pi/2$

c. $\tan^{-1}\sqrt{3}$

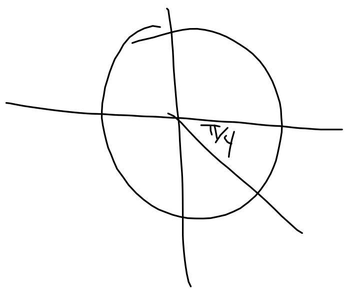
$\pi/3$

d. $\csc^{-1}\left(-\sqrt{2}\right)$

$= -\pi/4$

$\csc\theta = -\sqrt{2}$

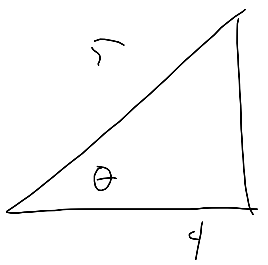
$\sin\theta = -\frac{\sqrt{2}}{2}$



Example 2) Evaluate the following. Make a picture to describe the situation.

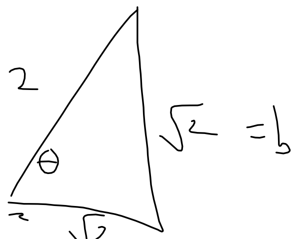
a. $\sin\left(\arctan\frac{3}{4}\right) = 3/5$

$\tan\theta = 3/4$



b. $\tan\left(\arccos\frac{\sqrt{2}}{2}\right) = 1$

$\cos\theta = \sqrt{2}/2$

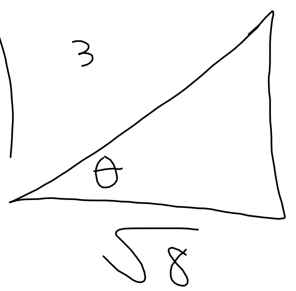


$c^2 = a^2 + b^2$

$9 = (\sqrt{2})^2 + b^2$

c) $\sec\left(\arcsin \frac{1}{3}\right) = \frac{3}{\sqrt{8}}$ $\qquad \frac{2}{-5}$

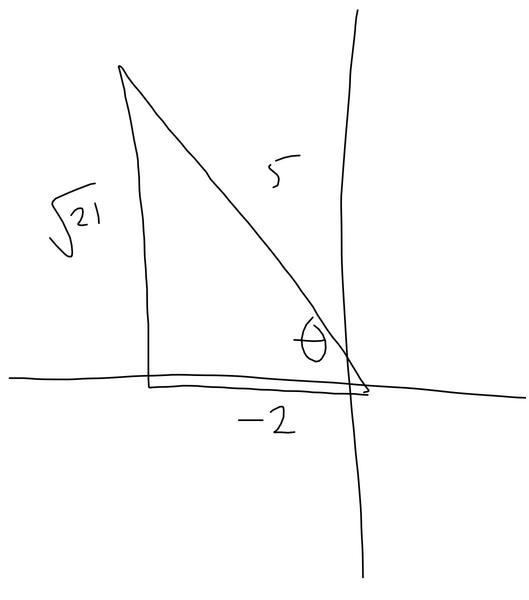
$\cot\left(\arccos -\left(\frac{2}{5}\right)\right)$



$c^2 = a^2 + b^2$

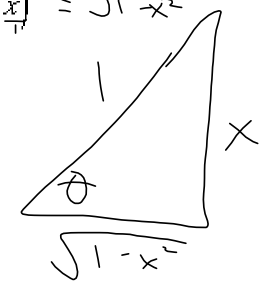
d) $\cot\left(\arccos\left(-\frac{2}{5}\right)\right)$

$= -2/\sqrt{21}$

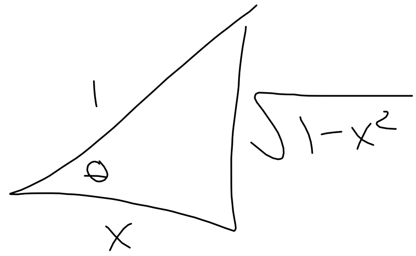


Example 3) Evaluate the following. Make a picture to describe the situation.

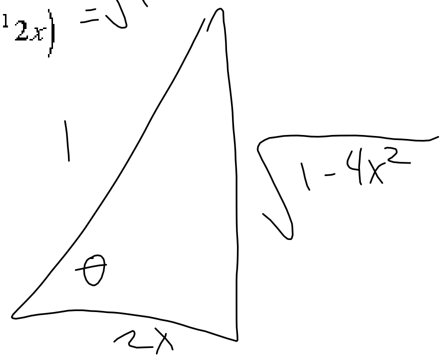
a. $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$



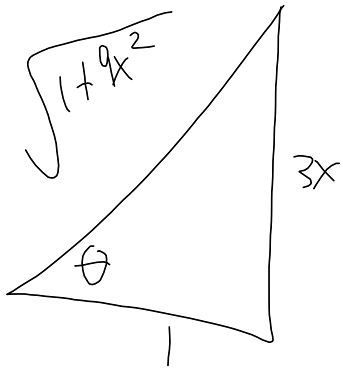
b. $\tan\left(\cos^{-1} x\right) = \dfrac{\sqrt{1-x^2}}{x}$



c. $\sin\left(\cos^{-1} 2x\right) = \sqrt{1-4x^2}$

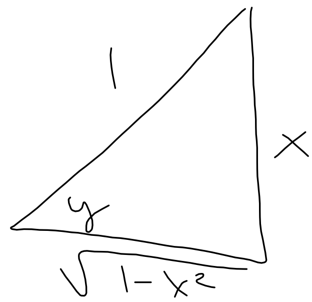


d. $\sin\left(\tan^{-1} 3x\right) = \dfrac{3x}{\sqrt{1+9x^2}}$



$$\frac{d}{dx}\left(\underbrace{\sin^{-1} x}_{y}\right)$$

$$\sin y = x$$



$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

So now we can take derivatives of inverse trig functions. Find $\frac{d}{dx}\left(\sin^{-1} x\right)$

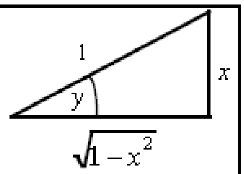
$$y = \sin^{-1} x \quad \Leftrightarrow \quad \sin y = x.$$

$y = \sin^{-1} x \Leftarrow \sin y = x$

Draw a picture

the angle is y, **opposite** $= 1$, **hypotenuse** $= 1$

Remaining side is $\sqrt{1-x^2}$



Since $\sin y = x$, take the derivative of each side

$$\cos y \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

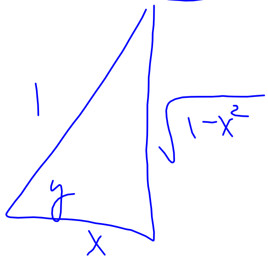
Eaxmple 4) Take the derivative of

a. $y = \cos^{-1} x$

$\cos y = x$

$-\sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-x^2}}$



b. $y = \tan^{-1} x$

The derivatives of the three inverse trig functions are as follows:

$$\frac{d}{dx}\left(\sin^{-1} u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cos^{-1} u\right) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\tan^{-1} u\right) = \frac{1}{1+u^2}\frac{du}{dx}$$

Example 5) Find the derivatives of

a. $y = \sin^{-1} 4x$

$B = 4x$

$M = \sin^{-1} x$

$$y' = \frac{4}{\sqrt{1-(4x)^2}}$$

b. $y = \tan^{-1} x^3$

$B = x^3$

$M = \tan^{-1} x$

$$y' = \frac{3x^2}{1+x^6}$$

c. $y = \left(\cos^{-1}\frac{x}{2}\right)^3$

$B = \cos^{-1}\frac{x}{2}$

$M = x^3$

d. $y = x\sin^{-1}x + \sqrt{1-x^2}$

$y' = 3\left(\cos^{-1}\frac{x}{2}\right)^2 \cdot \frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$

$B = \frac{x}{2}$

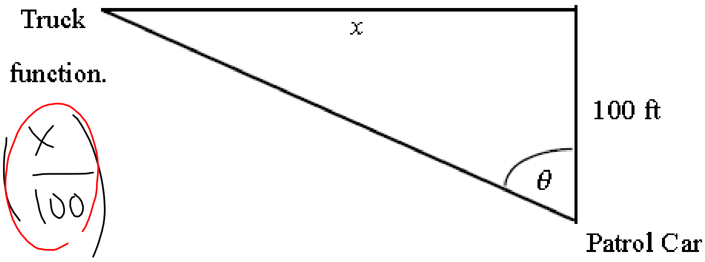
$M = \cos^{-1}x$

$= 3\left(\cos^{-1}\frac{x}{2}\right)^2 \cdot \frac{-1/2}{\sqrt{1-x^2/4}}$

d) $y' = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{\cancel{1}}{\cancel{2}}\left(1-x^2\right)^{-1/2} \cdot \left(-\cancel{2}x\right)$

$= \sin^{-1}x$

Example 6) An officer in a patrol car sitting 100 feet from the highway observes a truck approaching. At a particular instant t seconds, the truck is x feet down the road. The line of sight to the truck makes an angle of θ radians to a perpendicular line to the road.



a. Express θ as an inverse trig function.

$$\theta = \arctan\left(\frac{x}{100}\right)$$

b. Find $\frac{d\theta}{dt}$

c. When the truck is at $x =$ 500 ft, the angle is observed to be changing at a rate $\frac{d\theta}{dt}$ = -2 degrees/sec. How fast is the car going in ft/sec and mph?

$$\frac{d\theta}{dt} = \frac{1}{1 + \frac{x^2}{100^2}} \cdot \frac{1}{100} \frac{dx}{dt}$$

$$-\pi/90 = \frac{1}{1 + \frac{500^2}{100^2}} \cdot \frac{1}{100} \frac{dx}{dt}$$

$$2^\circ \frac{\pi}{180}$$